

The consistency equation hierarchy in single-field inflation models

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Inflationary consistency equations relate the scalar and tensor perturbations. We elucidate the infinite hierarchy of consistency equations of single-field inflation, the first of which is the well-known relation $A_T^2/A_S^2 = -n_T/2$ between the amplitudes and the tensor spectral index. We write a general expression for all consistency equations both to first and second-order in the slow-roll expansion. We discuss the relation to other consistency equations that have appeared in the literature, in particular demonstrating that the approximate consistency equation recently introduced by Chung and collaborators is equivalent to the second consistency equation of Lidsey et al. (1997).

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I. INTRODUCTION

An important prediction of the simplest inflationary models, driven by a single canonically-normalized scalar field, is that there should be relations between the spectra of scalar and tensor perturbations. The simplest such relation, usually referred to as the *consistency relation*, employs the slow-roll approximation and relates the relative amplitude of the tensor and scalar power spectra to the tensor spectral index. Such a specific relationship, if verified experimentally, would give powerful support to the single-field inflationary paradigm.

In this article, we point out that this consistency relation is the first of an infinite hierarchy of consistency relations, connecting ever higher derivatives of the spectra. This hierarchy exists even at lowest-order in the slow-roll approximation. That such a hierarchy exists was first noted in the review of Lidsey et al. [1], but we give here for the first time explicit expressions for these relations, both at lowest-order and next-order in slow-roll. Our analysis is restricted to the simplest class of inflation models, namely single-field slow-roll inflation with general relativity assumed valid.

To some extent this exercise is an academic one, as there seems little prospect of testing any of these relations beyond the first, and even it is likely to prove challenging [2]. Nevertheless, these relations offer a complete account of the connections between the two spectra, and so any other claimed consistency relation, exact or approximate, must follow from them if they are indeed consistency relations. In particular we examine the relationship between our formalism and the approximate consistency relation introduced by Chung, Shiu and Trodden [3] and further explored by Chung and Romano [4]. We demonstrate that it is indeed equivalent to the second consistency equation in the hierarchy, as already given by Lidsey et al. [1].

II. DEFINITIONS

Following the notation of Lidsey et al. [1], the spectra of scalar and tensor modes can be written

$$A_S(k) \cong \frac{4}{5m_{\text{Pl}}^2} [1 - (2C + 1)\epsilon + C\eta] \left. \frac{H^2}{|H'|} \right|_{k=aH}; \quad (1)$$

$$A_T(k) \cong \frac{2}{5\sqrt{\pi}} [1 - (C + 1)\epsilon] \left. \frac{H}{m_{\text{Pl}}} \right|_{k=aH}. \quad (2)$$

Here H is the Hubble parameter, prime is derivative with respect to field value ϕ , and $C \simeq -0.73$ is a constant. The terms in square brackets are the Stewart–Lyth slow-roll correction to the spectrum [5]; setting the square brackets to one gives the slow-roll result. We will use the symbol ‘ \cong ’ to indicate expressions as being equal within the slow-roll approximation to the order indicated by the included terms. The first few slow-roll parameters are defined by [6]

$$\epsilon \equiv \frac{m_{\text{Pl}}^2}{4\pi} \left(\frac{H'}{H} \right)^2; \quad (3)$$

$$\eta \equiv \frac{m_{\text{Pl}}^2}{4\pi} \frac{H''}{H}; \quad (4)$$

$$\xi \equiv \frac{m_{\text{Pl}}^2}{4\pi} \left(\frac{H' H'''}{H^2} \right)^{1/2}; \quad (5)$$

$$\sigma \equiv \frac{m_{\text{Pl}}^2}{4\pi} \left(\frac{H'^2 H''''}{H^3} \right)^{1/3}. \quad (6)$$

The wavenumber k can be related to the scalar field value via the exact relation

$$\frac{d \ln k}{d\phi} = \frac{4\pi}{m_{\text{Pl}}^2} \frac{H}{H'} (\epsilon - 1), \quad (7)$$

where without loss of generality we have assumed ϕ to increase during inflation.

The spectral indices are defined by

$$n_S - 1 \equiv \frac{d \ln A_S^2}{d \ln k}; \quad n_T \equiv \frac{d \ln A_T^2}{d \ln k}. \quad (8)$$

They and their derivatives can be written in terms of the slow-roll parameters by expressions such as

$$n_S - 1 \cong -4\epsilon + 2\eta + \left[-(8C + 8)\epsilon^2 + (6 + 10C)\epsilon\eta - 2C\xi^2 \right]; \quad (9)$$

$$n_T \cong -2\epsilon + \left[-(6 + 4C)\epsilon^2 + (4 + 4C)\epsilon\eta \right]; \quad (10)$$

$$\begin{aligned} \frac{dn_S}{d \ln k} \cong & -8\epsilon^2 + 10\epsilon\eta - 2\xi^2 + \\ & \left[-(40 + 32C)\epsilon^3 + (60 + 62C)\epsilon^2\eta \right. \\ & \left. - (12 + 20C)\epsilon\eta^2 - (8 + 14C)\epsilon\xi^2 \right. \\ & \left. + 2C\eta\xi^2 + 2C\sigma^3 \right]; \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{dn_T}{d \ln k} \cong & -4\epsilon^2 + 4\epsilon\eta + \\ & \left[-(28 + 16C)\epsilon^3 + (40 + 28C)\epsilon^2\eta \right. \\ & \left. - (8 + 8C)\epsilon\eta^2 - (4 + 4C)\epsilon\xi^2 \right]. \end{aligned} \quad (12)$$

In each case the term enclosed in square brackets is higher order in the slow-roll expansion, and is omitted when discussing lowest-order results.

III. THE CONSISTENCY EQUATION HIERARCHY: LOWEST-ORDER IN SLOW-ROLL

In this section we restrict ourselves to the slow-roll case, setting the square brackets in Eqs. (1) and (2) equal to one. Some simple algebra immediately leads to the standard consistency equation

$$2 \frac{A_T^2}{A_S^2} \cong -n_T \quad (13)$$

Note that n_T is always negative by definition. This relation was implicit in the results of Ref. [7], which was the first to write down the full slow-roll expressions, and was made explicit and named the consistency equation in Ref. [8].

Although this is the standard form of the relation (sometimes with a different coefficient if the spectra are defined with a different normalization), it somewhat conceals the physical underpinning of the consistency equation, which is that the two functions $A_S(k)$ and $A_T(k)$ have a common origin in the single function $V(\phi)$, and hence must be related through elimination of $V(\phi)$ from their defining equations. This is more explicit if we write all the scalar terms on one side and all the tensor ones on the other, to obtain

$$A_S^2 \cong -2 \frac{A_T^2}{n_T}. \quad (14)$$

It is clear from this expression that specifying the tensors completely defines the physical situation, and the corresponding scalar spectrum can be uniquely obtained from the consistency relation. If instead the scalars are specified, however, this is a differential equation for the

tensors whose solution yields a one-parameter set of physical models giving that scalar spectrum and each obeying the consistency equation.

The above equation is usually assumed to hold at one particular scale, often combined with the somewhat inconsistent assumption that the spectra are power-laws with different spectral indices ($n_S - 1 \neq n_T$). However further consistency relations can be obtained, as first shown in Ref. [1], by realizing that the consistency equation is supposed to hold on all scales. One can proceed, for instance, by Taylor expanding both sides of Eq. (14) in $\ln k$ about some characteristic scale k_0 ,¹ giving

$$\begin{aligned} A_S^2 + \frac{dA_S^2}{d \ln k} \ln \frac{k}{k_0} + \frac{1}{2} \frac{d^2 A_S^2}{d \ln k^2} \ln^2 \frac{k}{k_0} + \dots \cong \\ -2 \frac{A_T^2}{n_T} + \frac{d[-2 \frac{A_T^2}{n_T}]}{d \ln k} \ln \frac{k}{k_0} + \frac{1}{2} \frac{d^2[-2 \frac{A_T^2}{n_T}]}{d \ln k^2} \ln^2 \frac{k}{k_0} + \dots \end{aligned} \quad (15)$$

where the expansion coefficients are all evaluated at k_0 . Equating the coefficients on each side gives

$$\frac{d^{(i)} A_S^2}{d \ln k^{(i)}} \cong \frac{d^{(i)} [-2 \frac{A_T^2}{n_T}]}{d \ln k^{(i)}}, \quad i = 0, 1, \dots, \quad (16)$$

with both sides evaluated at some arbitrary scale k_0 . This represents the generic form of an infinite hierarchy of consistency equations.

The first derivative, $i = 1$, gives the lowest-order version of the second consistency equation

$$\frac{dn_T}{d \ln k} \cong 2 \frac{A_T^2}{A_S^2} \left[2 \frac{A_T^2}{A_S^2} + (n_S - 1) \right]. \quad (17)$$

$$\cong n_T [n_T - (n_S - 1)]. \quad (18)$$

This equation first appeared in Ref. [9] without being explicitly recognized as a consistency equation, that role being pointed out in Ref. [1]. Eq. (16) is the first time an explicit form for the full infinite hierarchy has been written down.

It is possible to rewrite Eq. (16) in an interesting alternate form using only the spectral indices

$$\frac{d^{(i-1)}(n_S - 1)}{d \ln k^{(i-1)}} \cong \frac{d^{(i-1)} n_T}{d \ln k^{(i-1)}} - \frac{d^{(i)} \ln(-n_T)}{d \ln k^{(i)}}, \quad i = 1, 2, \dots. \quad (19)$$

This does not encode the normal (first) consistency relation, but does capture all the others in quite an elegant form.

IV. THE CONSISTENCY EQUATION HIERARCHY: NEXT-ORDER IN SLOW-ROLL

All of the above can readily be generalized to next-order in slow roll by retaining the full form of Eqs. (1)

¹ In carrying out these manipulations, note that the order-by-order slow-roll expansion is preserved both by taking derivatives and logarithms.

and (2). The next order of the first consistency equation was first given in Ref. [10], and quoted in Ref. [1] as

$$n_T \cong -2 \frac{A_T^2}{A_S^2} \left[1 - \frac{A_T^2}{A_S^2} - (n_S - 1) \right]. \quad (20)$$

In order to separate the scalar quantities in this expression from the tensor ones, we write it as

$$-\frac{A_T^2}{A_S^2} \frac{2}{n_T} \cong 1 - \frac{1}{2} n_T + (n_S - 1) \quad (21)$$

and use small-parameter manipulations to obtain

$$A_S^2 [1 + (n_S - 1)] \cong -\frac{2A_T^2}{n_T} \left[1 + \frac{1}{2} n_T \right], \quad (22)$$

where the scalars all stand to the left and the tensors to the right. Note that the tensors no longer uniquely specify the scalars, though the requirement of a subdominant next-order term (for the expansion to make sense) will give a practically-unique scalar spectrum for a given tensor one.

The hierarchy of consistency equations to next-order, with scalars and tensors separated, is obtained by differentiating Eq. (22) repeatedly with respect to $\ln k$. For instance, we can take Eq. (18) to next order by differentiating Eq. (20) once to get

$$\begin{aligned} \frac{dn_T}{d \ln k} &\cong n_T [n_T - (n_S - 1)] \\ &+ n_T \left[\frac{n_T}{2} (n_T - (n_S - 1)) - \frac{dn_S}{d \ln k} \right]. \end{aligned} \quad (23)$$

The first term on the right-hand side is of course the first-order version of the second consistency equation.

V. RELATION TO APPROXIMATE CONSISTENCY EQUATIONS

Since Eq. (16) and its higher-order equivalents give a complete account of relations between the scalar and tensor spectra, any other consistency relations claimed in the literature, approximate or otherwise, must follow from them. One such is a relation proposed by Chung, Shiu, and Trodden [3] and explored in detail by Chung and Romano [4], concerning a near coincidence of scales in models with strong running. Another appears in Lidsey and Tavakol [11] under the assumption of constant running. We examine each in turn.

A. Coincidence of scales

The authors of Refs. [3, 4] note that in models with large running, there is an approximate coincidence of the scales where $n_S - 1 = 0$ and where the tensor-to-scalar

ratio reaches a minimum. The first of those scales is denoted k_1 , and the second k_2 . By definition²

$$\left. \frac{d \ln(A_T^2/A_S^2)}{d \ln k} \right|_{k_2} = 0 \implies n_S(k_2) - 1 = n_T(k_2). \quad (24)$$

Since the two conditions equate n_S to different values, the relation is clearly not exact. The difference between the two scales can be defined as $\Delta N = \ln k_2/k_1$. If we assume that the runnings are constant, but make no assumption that the spectra arise from inflation, there is a general expression

$$\Delta N = \frac{(n_S - 1) - n_T}{dn_T/d \ln k - dn_S/d \ln k} + \frac{n_S - 1}{dn_S/d \ln k} \quad (25)$$

where the observables are evaluated at an arbitrary scale k_0 . If we further specialize that the expansion scale is chosen to be one of the scales k_1 or k_2 (bearing in mind that before the fit to the data we wouldn't know where those scales are, and that they may not lie where the data is), this expression simplifies to

$$k_1 : \quad \Delta N = -\frac{n_T}{dn_T/d \ln k - dn_S/d \ln k}; \quad (26)$$

$$k_2 : \quad \Delta N = \frac{n_S - 1}{dn_S/d \ln k}. \quad (27)$$

Two things to note about these equations are as follows. Firstly, slow-roll inflation predicts that the two scales are far apart, not close, since the denominator is one order higher in slow-roll than the numerator and hence $\Delta N \sim \mathcal{O}(1/\epsilon)$ in the absence of cancellations. If they are close, partial cancellations will have allowed the running to be large while the scalar spectral index remains close to unity (this can happen plausibly, for instance, in running-mass inflation models [12]). Secondly, the above relations are *not* consistency relations, as no inflationary input has been added and they are true of arbitrary spectra, not just those tied together as inflation predicts. In particular, if n_S is already measured at k_2 , then measuring ΔN and measuring $dn_S/d \ln k$ at k_2 are the same thing.

The above equations can be converted into consistency equations by substitution of the inflationary spectra, thus enforcing the relation between tensor and scalars. For instance, doing this in Eq. (26) to lowest order yields

$$\Delta N \cong \frac{\epsilon}{\xi^2 - 4\epsilon^2}, \quad (28)$$

where the slow-roll parameters are evaluated at k_1 . This is precisely Eq. (151) of Ref. [3] rewritten in our notation. Carrying out the same procedure to second-order

² In parts of their paper, Chung and Romano define scale k_2 as being where ϵ reaches its extremum. Beyond the slow-roll approximation this is not quite equivalent to our definition, which we believe is more appropriate since ϵ is not a direct observable.

in Eq. (27) yields Eq. (21) of Ref. [4] (note that their definition of the constant C is different to ours).

That these relations are equivalent to the consistency equations, specifically the second one given by Eq. (18) or Eq. (23), is rather subtle. Now, Eq. (28) is not actually a useful form, since ϵ and ξ are not directly observable. Sufficient observables to determine them are n_T and $dn_S/d\ln k$, bearing in mind that by definition $n_S = 1$ at the scale k_1 where their relation applies (and hence $2\epsilon \cong \eta$ to the required order). This allows us to rewrite as

$$\Delta N \cong -\frac{n_T}{n_T^2 - dn_S/d\ln k}. \quad (29)$$

Their test therefore proposes to measure the three quantities in this expression and verify that this relation holds.

However we know that the general expression for ΔN is Eq. (26). Comparing with Eq. (29), we see that their test actually seeks to confirm that

$$\frac{dn_T}{d\ln k} \cong n_T^2. \quad (30)$$

This is nothing other than the second consistency equation, Eq. (18), evaluated at k_1 so that $n_S - 1$ vanishes. Transforming to any other scale would then give the full version of the (lowest-order) second consistency equation.

In conclusion, while it appears that their method does not measure $dn_T/d\ln k$, in fact the measurement of ΔN along with the other observables does so implicitly, and their test is precisely equivalent to the second consistency equation, the lowest-order version of which was already given in Ref. [1].

B. Constant running

A different relation, advertised as independent of the inflationary potential, was given by Lidsey and Tavakol [11]. They noted that if it were assumed that the scalar running is constant, then the (lowest-order) equation for it, Eq. (11) with the square bracket set to zero, can be written in terms of the scalar spectral index, the tensor-to-scalar ratio, and an undetermined constant \tilde{c} , eliminating the dependence on the potential. Their Eq. (18) reads

$$\frac{A_S^2}{A_T^2} \exp \left[-\frac{(n_S - 1)^2}{2dn_S/d\ln k} \right] - \left(\frac{2\pi}{dn_S/d\ln k} \right)^{1/2} \operatorname{erf} \left[\frac{n_S - 1}{\sqrt{2dn_S/d\ln k}} \right] = \tilde{c}. \quad (31)$$

As they acknowledge, in the usual interpretation where the observables are given at a fixed (though arbitrary) expansion scale, this is not a consistency equation as determining \tilde{c} is equivalent to determining $dn_S/d\ln k$. It

is further evident that it is not a consistency equation since it does not mention the tensor spectral index or its derivatives, whereas all members of our consistency equation hierarchy, an exhaustive list of relations between observables, do feature those.

They suggest that the equation can be given content by evaluating it at two different scales, the first used to fix \tilde{c} and the second to test the relation. However this appears primarily to be a test of the assumption of constant running, with the implications for inflationary dynamics depending on the details of how that assumption might fail — typical inflation models do predict some deviation from constant running. In any event, their relation does not follow from the consistency equation hierarchy we have described.

VI. CONCLUSIONS

Single-field inflation predicts not just one consistency relation, but an infinite hierarchy, each of which can be considered at different orders in the slow-roll expansion [1]. We have for the first time written down explicit expressions for all these relations, and shown how they relate to other consistency equations found in the literature. Observed violation of these consistency relations would exclude single-field slow-roll inflation under Einstein gravity, pointing instead perhaps to multi-field phenomena, non-Einsteinian gravity, or a non-inflationary origin of perturbations.

It is difficult to be optimistic about attempts to test any other than the lowest-order version of the first consistency equation, the famous $A_T^2/A_S^2 = -n_T/2$ relation, which itself is quite challenging. Song and Knox [2] have made a comprehensive study of the ability of cosmic microwave background experiments to test this consistency relation. They also discuss taking that relation to next order; doing so introduces an extra observable n_S , which should be accurately measurable, but current observational constraints already place us in a parameter regime where the next-order correction should be too small to observe due to the expected observational uncertainty on n_T . Going instead to the lowest-order version of the second consistency relation, Eq. (17), introduces the distinctly challenging observable $dn_T/d\ln k$. This observable is also required to meaningfully test the coincidence of scales described in Refs. [3, 4], which we have shown is equivalent to our results and indeed those given in Ref. [1].

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